

A Fully Redundant Double Difference Algorithm for Obtaining Minimum Variance Estimates From GPS Observations

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December 15, 1986



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Space Administration

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The research described in this publication was carried out by the Jet Propulsion Laboratory, California Institute of Technology, under a contract with the National Aeronautics and Space Administration.

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ABSTRACT

In double differencing a regression system obtained from concurrent Global Positioning System (GPS) observation sequences, one either under-samples the system to avoid introducing colored measurement statistics, or one fully samples the system incurring the resulting non-diagonal covariance matrix for the differenced measurement errors. A suboptimal estimation result will be obtained in the under-sampling case and will also be obtained in the fully sampled case unless the color noise statistics are taken into account. The latter approach requires a least squares weighting matrix derived from inversion of a non-diagonal covariance matrix for the differenced measurement errors instead of inversion of the customary diagonal one associated with white noise processes. This publication presents the so-called fully redundant double differencing algorithm for generating a weighted double differenced regression system that yields equivalent estimation results, but features for certain cases a diagonal weighting matrix even though the differenced measurement error statistics are highly colored.

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I. INTRODUCTION

The concurrent operation of two or more Global Positioning System (GPS) ground terminals observing multiple GPS satellites generates multiple data streams that yield high accuracy relative positioning. Concurrent operations produce simultaneous observational epoch sequences among the multiple data streams, which allows one to eliminate errors in the state vector estimates that arise from common mode-error sources in the data streams. In particular, the effects of unknown and imperfectly running clocks on board the satellites and in the ground terminals may be avoided. Although a parameterized representation of the clock errors in the regression equations may be estimated in parallel with the state vector in an expanded state space mode, this approach leads to very large arrays unless a partitioning technique is used. Alternative approaches eliminate the clock terms before obtaining the estimate of the state vector. In the latter approach, the regression equations are double differenced at simultaneous epochs to cancel the clock parameters before the state vector is estimated. Alternatively, the regression equations are not differenced; rather, a series of orthogonal transformations are applied to the system. These transformations effectively eliminate these common mode errors while yielding a minimum variance estimate of the state vector. One technique following the latter approach involves the operation on the regression system in either a batch or sequential mode by a series of Householder matrices, tailored to this problem, that transforms the information matrix into the form of an upper triangular array. The lower portion of this upper array is explicitly free of clock information. Thus, these Householder transformations effectively eliminate clock errors before the actual inversion of the state vector triangularized information matrix takes place [1,2]. The advantage of the Householder matrices or similar approaches is that additional modeling and a priori information about these errors can be readily introduced in the estimation process without explicitly solving for the clock errors. For example, the stochastic properties of the clock variability may be modeled by specification of the parameters from a first order Markov process. Alternatively, the clocks in GPS terminals collocated at VLBI fiducial sites can be slaved to the resident hydrogen maser. Their near-linear time behavior provides important additional information, particularly in the control of errors in the estimated GPS ephemerides, which would be lost in a double differencing mode.

The double differencing approach appears to offer simplicity by eliminating the clock parameters straightaway. Double differencing is attractive because the differenced residuals lend themselves to an easy and traceable physical interpretation, which is not usually the case with residuals obtained from Householder or similar orthogonal transformations. Double differencing is also particularly useful in a noisy environment for editing functions, such as detecting and removing carrier cycle dropouts and other error sources from the data streams. A potential disadvantage of double differencing is that it usually results in colored noise in the estimation problem. Also, techniques that involve an orthogonalization of the information matrix such as, for example, the Householder approach, tend to be more computationally efficient. Nevertheless, there are situations where double differencing may be a practical necessity or even preferable in a data management sense.

This publication addresses two full-rank approaches to double differencing: the so-called minimally redundant approach and the fully redundant approach. A full-rank approach is one in which the double differencing in a matrix operator sense has maximal rank and hence spans the linear vector space defined by the regression system. Double differencing introduces colored noise statistics into the differenced measurement error covariance. To obtain estimation results that are equivalent to those obtained by the expanded state or Householder methods, these approaches require a least-squares weighting matrix that is derived either from an inversion of the colored-measurement error-covariance matrix, which is nondiagonal to a varying degree depending on the differencing scheme used and the number of stations and satellites involved. Alternatively, one can use a whitening transformation on the differenced regression system before obtaining the inverse.

The computational time for numerical inversion of the nondiagonal covariance matrix tends to grow as the cube of its size. However, if the undifferenced measurement statistics are white, then the double-differenced covariance matrix may be only mildly striped if either the number of terminals or satellites is moderate. In this case, more efficient matrix inversion techniques are available [3], which tend to grow somewhat more slowly than the cube of the matrix size. Also, analytic inverses are available for certain measurement error covariances with simple forms. The fully redundant approach features a diagonal weighting matrix for certain forms for the undifferenced

measurement-error covariance even though the double-difference measurement-error statistics are highly colored. This feature may provide a computational advantage in limited situations. On the other hand, the number of differencing operations themselves in this approach tends to grow quadratically with the number of terminals and with the number of GPS satellites. Comparative studies indicate that this method should be considered competitive in computational time only in those cases where the number of terminals or the number of observed satellites is modest, i.e., roughly four or less, and only in cases where analytic forms for the diagonal weighting matrix are available. If analytical forms are not available and one wishes to use double differencing, then a full-rank but minimally redundant doubled-differencing approach appears to be the most straightforward [3].

After a general description of the regression problem given in the next section, this publication discusses double differencing operations and the equivalence under certain conditions of the resulting least-squares estimates to those obtained by other techniques. The publication then discusses minimally and fully redundant double difference matrix operators and their least-squares weighting matrices. In the fully redundant approach, its corresponding white-noise-equivalent diagonal weighting matrix is developed. It is shown that analytic forms of this diagonal matrix can be generated in some cases. In theory a composite diagonal and antisymmetric weighting matrix can be generated in all cases, but the practicability of generating analytic forms appears limited to certain restrictive assumptions about the topological properties of the undifferenced measurement error covariance. Finally, the paper treats the problem of missing data streams or outages in double differencing.

II. PROBLEM STATEMENT

We consider the case where m ground terminals are concurrently tracking r GPS satellites from which mr distinct data streams are generated in parallel with simultaneous observational epoch sequences in each stream. The measurement types assumed here are either the undifferenced carrier phase or the group delay based ranging, which are characteristic of single GPS receiver operations. The carrier phase measurements are assumed to be phase connected although this assumption is not fundamental to what follows. In addition to depending on the state vector, the measurements are corrupted by the offsets of the clocks from universal time. A linear regression system of the form

$$y = \mathcal{A} X + e = [B:H] \begin{bmatrix} b \\ x \end{bmatrix} + e \quad (1)$$

is assumed where the vectors y and X represent small differences from nominal values and where:

y is the mrN by 1 observation vector

N is the number of observational epochs in each data stream

\mathcal{A} is the grand information matrix for the regression system

X is the expanded state vector of dimension $p + (m + r)N$ by 1

e is the measurement error vector

B is the mrN by $(m + r)N$ clock vector information matrix

H is the mrN by p state vector information matrix

b is the $(m + r)N$ by 1 clock error parameter vector

x is the p by 1 state vector

The state vector consists of the parameter set except clocks that are required to properly define the dynamical and observational systems. Thus, in addition to state vector information for GPS terminals and GPS satellites, x would also include carrier cycle ambiguities, propagation media parameters, etc.

The clock characteristics aboard each satellite and in each terminal are not modeled in this problem. Rather, each clock error at any epoch is assumed (rather pessimistically) to be uncorrelated with any other. Thus, the vector b represents the $(m + r)N$ separate clock errors. The rank of B is $(m + r - 1)N$ because the sum of the clock parameters at each epoch is unobservable for the linear problem. The information matrix H is assumed to be of rank p . The error vector e is assumed to have standard mean zero white Gaussian noise properties and a covariance matrix of Λ_e .

It is assumed that y and \mathcal{A} have been appropriately formatted, with the time-ordered subblock of observation residuals and regression coefficients generated from observations of the same satellite by the same terminal grouped in contiguous rows. Additionally, the m subblocks arising from observations of the same satellite by the m different terminals are placed contiguously to create a single block containing the information from the entire set of observations of the same satellite. The r blocks associated with observations of the r different satellites, each with the identical substructure, are grouped contiguously to fill out the complete observation vector and information matrix. For example, with $r = 4$ and $m = 3$, H would appear as

$$H^T = \begin{matrix} \underbrace{\hspace{1.5cm}}_{m = 3} \\ \left[\begin{array}{cccccccccccccccc} A_1^T & A_2^T & A_3^T & B_1^T & B_2^T & B_3^T & C_1^T & C_2^T & C_3^T & D_1^T & D_2^T & D_3^T \end{array} \right] \\ \underbrace{\hspace{1.5cm}}_{r = 4} \end{matrix} \quad (2)$$

where the subscript denotes the terminal and the letters A, B, C and D denote the information submatrices from N observations each from four different satellites, denoting subscripts as terminal designations and superscripts as satellite designations, then, following the form in Eq. (2), H may be written as

$$H^T = \begin{matrix} \underbrace{\hspace{1.5cm}}_{m \text{ subblocks}} \\ \left[\begin{array}{cccccccc} T_1 & & T_1 & & & & T_r & T_r \\ H_1^1 & \dots & H_m^1 & & & & H_1^r & \dots & H_m^r \end{array} \right] \\ \underbrace{\hspace{1.5cm}}_{r \text{ blocks}} \end{matrix} \quad (3)$$

The units and scale of the clock parameters have been chosen the same as those of the observation vector y . It follows, provided that the regression equations have been formatted according to Eq. (3), that B , the clock parameter information matrix, consists of ones and zeros in the pattern shown below

$$B = \left[\begin{array}{ccc|ccc} I_m & -1_m & 0 & & & 0 \\ \hline I_m & 0 & -1_m & & & 0 \\ \hline \cdot & & & \cdot & & \\ \cdot & & & & \cdot & \\ \cdot & & & & & \cdot \\ \hline I_m & 0 & 0 & & & -1_m \end{array} \right] \left. \vphantom{\begin{array}{ccc|ccc} \end{array}} \right\} \begin{array}{l} mN \\ rN \end{array} \quad (4)$$

where

$$I_m = \left[\begin{array}{ccc|ccc} I & 0 & & & & 0 \\ \hline 0 & I & & & & 0 \\ \hline & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ \hline 0 & 0 & & & & I \end{array} \right] \left. \vphantom{\begin{array}{ccc|ccc} \end{array}} \right\} mN \quad (5)$$

and 1_m is given by

$$1_m^T = \left[\overbrace{I : \cdots : I}^m \right] \quad (6)$$

The identity matrices in Eqs. (5) and (6) are N by N .

In a batch mode version of the Householder approach for eliminating the clock parameters [1], one operates on Eq. (1) with a series of orthogonal Householder matrices [2] that transforms it into the form

$$\begin{bmatrix} y_b \\ -y_x \end{bmatrix} = R \begin{bmatrix} b \\ x \end{bmatrix} = \begin{bmatrix} R_b & R_{bx} \\ 0 & R_x \end{bmatrix} \begin{bmatrix} b \\ x \end{bmatrix} \quad (7)$$

where R is a square $(m+r+p)N$ matrix consisting of an upper triangular array with zeros below the diagonal. In particular, the minimum variance state vector estimate \hat{x} is given by

$$\hat{x} = R_x^{-1} y_x \quad (8)$$

and its covariance matrix $\Lambda_{\hat{x}}$, is given by

$$\Lambda_{\hat{x}} = R_x^{-1} (R_x^T)^{-1} \quad (9)$$

It is noted that R_x is the transformed information matrix in which the clock information is explicitly absent; thus, there is no need to jointly solve for clock parameters unless one desires their estimates or has other a priori information about them.

A weighted least squares method that is equivalent to but less efficient than the Householder method is the so-called augmented state space approach. Because it is relatively better known we will use this formulation rather than the Householder formulation to establish the equivalency of the double differencing approach. Here, one adjoins the clock parameter vector b , to the state vector x , and obtains a joint minimum variance estimate of this augmented state vector using a standard least squares batch mode approach. The grand covariance matrix of the augmented state vector estimate is obtained from the pseudoinverse of

$$\left[\mathcal{A} \Lambda_e^{-1} \mathcal{A} \right] = \left[\begin{array}{c|c} H^T \Lambda_e^{-1} H & H^T \Lambda_e^{-1} B \\ \hline B^T \Lambda_e^{-1} H & B^T \Lambda_e^{-1} B \end{array} \right] \quad (10)$$

The pseudoinverse of Eq. (10) can be written in partition form as

$$\left[\mathcal{A} \Lambda_e^{-1} \mathcal{A} \right]^\dagger = \left[\begin{array}{c|c} \Lambda_{\hat{x}} & F \\ \hline F^T & G \end{array} \right] \quad (11)$$

where $\Lambda_{\hat{x}}$ is the covariance of the state vector estimate \hat{x} , and where F and G are the cross-covariance and covariance associated with the clock parameter sequences. (Pseudoinverses appear because the rank of $B^T B$ is $m+r-1$ instead of $m+r$. By fixing the clock at the first ground terminal, for example, one then eliminates the first column from B , thereby rendering $B^T B$ into a full rank matrix. In this case, the pseudoinverse operations in the subsequent discussions may be replaced by a standard inverse. The ensuing results are not materially different from those obtained with the pseudoinverse approach.) We assume that H is of full parameter rank p and that the column vectors of H are linearly independent of those in B . Using the properties of pseudomatrices, it can be shown for matrices of the form in Eq. (11) that

$$[H^T \Pi H] \Lambda_{\hat{x}}^{-1} [H^T \Pi H] = [H^T \Pi H] \quad (12)$$

where the matrix Π is defined by

$$\Pi = \Lambda_e^{-1} - \Lambda_e^{-1} B [B^T \Lambda_e^{-1} B]^\dagger B^T \Lambda_e^{-1} \quad (13)$$

Hence, if the inverse of $[H^T \Pi H]$ exists, it follows that covariance of \hat{x} is given by

$$\Lambda_{\hat{x}}^{-1} = H^T \Pi H \quad (14)$$

Also, it follows that \hat{x} is given by

$$\hat{x} = \Lambda_{\hat{x}}^{-1} H^T \Pi y \quad (15)$$

It can be shown [2] that Eqs.(8) and (15) are equivalent and that Eqs.(9) and (14) are also equivalent. Even in this augmented state space approach, by using this matrix partitioning technique one need not explicitly solve this system for the clock parameters.

Because the matrix Π and its factorization serves as a bridge between the augmented state space and the double differencing approaches, a few observations about some of its properties should be noted. From the least squares projection theorem in linear vector spaces [4] (or from multiplying Eq. (13) by B), it follows that Π is orthogonal to B . Also, Π , which has a rank of $(m-1)(r-1)N$, has the property that

$$\Pi = \Pi \Lambda_e \Pi \quad (16)$$

The matrix $\tilde{\Pi}$, given by

$$\tilde{\Pi} = \Lambda_e^{\frac{1}{2}} \Pi \Lambda_e^{\frac{1}{2}} \quad (17)$$

is an idempotent matrix and its own pseudoinverse with a trace equal to $(m-1)(r-1)N$.

Finally, it should be noted that Π itself, which is a relatively simple matrix to generate, serves as a matrix operator which, when applied to Eq. (1), transforms it into a new regression system that is explicitly free of clock terms. By premultiplying Eq. (1) by Π , one obtains

$$\Pi y = \Pi Hx + \Pi e \quad (18)$$

It is easily shown that the resulting least squares estimate for x and its covariance matrix obtained from Eq. (18) are formally identical to those in Eqs. (15) and (14). This suggests yet another method in lieu of double differencing for eliminating clock parameters.

III. DOUBLE DIFFERENCING OPERATIONS AND CONDITIONS FOR EQUIVALENCE

In the double differencing approach one operates on the system in Eq. (1) in such a way to explicitly eliminate the clock parameters from the resulting regression system before carrying out the filtering operation.

A double differencing operation first differences the regression equations associated with the simultaneous observations by two terminals of the same satellite, and then it differences two such first differences obtained from the same two terminals simultaneously observing two different satellites.

Alternatively, these operations may be reversed, but the result is the same.

While double differencing operations can be variously formulated, all formulations have the property of totally eliminating the clock matrix B from the resulting double differenced regression equations. In matrix notation, let \mathcal{D} be a double difference matrix operator. Then, it follows that

$$\mathcal{D}B = 0 \quad (19)$$

If one carries out the double differencing of the regression system in Eq. (1) following the \mathcal{D} -matrix formulation, one obtains

$$\mathcal{D}Y = \mathcal{D}Hx + \mathcal{D}e \quad (20)$$

a new system of regression equations to be solved in a least squares sense for the state vector x. Letting \tilde{e} be defined by

$$\tilde{e} = \mathcal{D}e \quad (21)$$

we have for the covariance of \tilde{e}

$$\Lambda_{\tilde{e}} = \mathcal{D} \Lambda_e \mathcal{D}^T \quad (22)$$

a nondiagonal, colored matrix whose size depends on one's choice for \mathcal{D} . The rank of $\Lambda_{\tilde{e}}$ is at most mrN , which may be smaller than its size depending on the specific differencing algorithm used. Consequently, one should use the pseudoinverse of $\Lambda_{\tilde{e}}$ in the formulation for the least squares estimate of x. Then, $\Lambda_{\hat{x}}$ is given by

$$\Lambda_{\hat{x}}^{-1} = H^T \left[\mathcal{D}^T (\mathcal{D} \Lambda_e \mathcal{D}^T)^{\dagger} \mathcal{D} \right] H \quad (23)$$

and \hat{x} is given by

$$\hat{x} = \Lambda_{\hat{x}}^{-1} H^T \left[\mathcal{D}^T (\mathcal{D} \Lambda_e \mathcal{D}^T)^{\dagger} \mathcal{D} \right] Y \quad (24)$$

If the state vector estimate and its covariance in Eqs. (24) and (23) are to be equivalent to those in Eqs. (15) and (14), it follows that

$$M \triangleq \mathcal{D}^T \left[\mathcal{D} \Lambda_e \mathcal{D}^T \right]^\dagger \mathcal{D} - \left[\Lambda_e^{-1} - \Lambda_e^{-1} B (B^T \Lambda_e^{-1} B)^{-1} B^T \Lambda_e^{-1} \right] \equiv 0 \quad (25)$$

To establish the conditions for this equivalence, it is sufficient to find mrN linearly independent nonzero vectors ξ such that

$$M\xi \equiv 0 \quad (26)$$

The matrix B contains $(m + r - 1)N$ linearly independent column vectors and from the orthogonality condition in Eq. (19) and the properties of pseudoinverse matrices it follows that

$$MB \equiv 0 \quad (27)$$

Next, we premultiply Eq. (25) by Λ_e (assumed to be full rank) and then by \mathcal{D} to obtain

$$\mathcal{D} \Lambda_e M \equiv 0 \quad (28)$$

If \mathcal{D} fully spans the vector space orthogonal to the space spanned by the columns of B , it will contain $(m - 1)(r - 1)N$ linearly independent column vectors. Because \mathcal{D} is also orthogonal to B , Eqs. (27) and (28) serve to establish that $M \equiv 0$ and, thus, to establish the equivalence of Eqs. (23) and (14), also Eqs. (24) and (15) when \mathcal{D} fully spans the space orthogonal to B . It is readily shown that any double difference operator that preserves the white noise statistics of the differenced measurement errors must have a rank less than $(m-1)(r-1)N$; hence, in general it will yield non-equivalent results.

An example of a full rank but minimally redundant operator is given by

$$\mathcal{D} = \left[\begin{array}{c|c|c|c|c} D & -D & 0 & \cdots & 0 \\ \hline D & 0 & -D & \cdots & 0 \\ \hline \cdot & & & & \\ \hline D & 0 & 0 & \cdots & -D \end{array} \right] \quad (m-1)(r-1) \text{ rows} \quad (29)$$

where D , a first difference operator of full rank $m-1$ but minimally redundant, is given by

$$D = \left[\begin{array}{cccccc} 1 & -1 & 0 & \cdot & \cdot & \cdot & 0 \\ 1 & 0 & -1 & \cdot & \cdot & \cdot & 0 \\ \cdot & & & & & & \\ \cdot & & & & & & \\ 1 & 0 & 0 & \cdot & \cdot & \cdot & -1 \end{array} \right] \left. \vphantom{\begin{array}{c} 1 \\ 1 \\ \cdot \\ \cdot \\ 1 \end{array}} \right\} (m-1) \text{ rows} \quad (30)$$

The operator in Eq. (29) is minimally redundant because it generates the least number of rows in the double differenced regression matrix while maintaining a full rank of $(m-1)(r-1)$. It is called "redundant" because one row and one block of the undifferenced system appear respectively in all rows and all blocks of the differenced system. The covariance of \tilde{e} (Eq. (22)) using this operator is non-diagonal and generally requires a numerical inversion to obtain the least squares weighting matrix. However, for modest values of m or r and/or for certain simple forms of Λ_e , analytic inverses can be readily derived. (See Appendix 1.) Remondi [3] has provided an example of this for $m = 2$.

IV. THE FULLY REDUNDANT DOUBLE DIFFERENCE MATRIX OPERATOR

Because the double differenced measurement error covariance matrix in Eq. (22) may be highly non-diagonal as a result of the differencing operations and because it may also be quite large, we present an algorithm which enables one to replace it with an equivalent diagonal form. To accomplish this we define a specific form for the double difference matrix operator. (See Appendix 2 for a more detailed discussion.)

Let us define a first difference operator D_k , which when applied to another matrix of k block rows, generates the $\binom{k}{2}$ distinct pair differences that can be formed from these block rows. For example, for $k = 1, 2, 3$, and 4 , D_k is given by

$$D_1 = 0, D_2 = [I \ : \ -I], D_3 = \left[\begin{array}{c|c|c} I & -I & 0 \\ \hline I & 0 & -I \\ \hline 0 & I & -I \end{array} \right], D_4 = \left[\begin{array}{c|c|c|c} I & -I & 0 & 0 \\ \hline I & 0 & -I & 0 \\ \hline I & 0 & 0 & -I \\ \hline 0 & I & -I & 0 \\ \hline 0 & I & 0 & -I \\ \hline 0 & 0 & I & -I \end{array} \right] \quad (31)$$

Thus, D_k operating on a k by p matrix H creates a new $\binom{k}{2}$ by p matrix, whose rows comprise the $\binom{k}{2}$ distinct ways that pair differences can be formed from the original k rows.

Next, the analogous double difference operator \mathcal{D} is defined by

$$\mathcal{D} = \left[\begin{array}{c} \left\{ \begin{array}{l} D - D \quad 0 \dots\dots\dots 0 \\ D \quad 0 - D \dots\dots\dots 0 \\ \vdots \\ D \quad 0 \quad 0 \dots\dots\dots -D \\ 0 \quad D - D \dots\dots\dots 0 \\ \vdots \\ 0 \quad D \quad 0 \dots\dots\dots -D \\ \vdots \\ 0 \quad 0 \quad 0 \dots\dots D - D \end{array} \right\} \begin{array}{l} r-1 \\ \\ \\ r-2 \\ \\ \\ 1 \end{array} \end{array} \right] \begin{array}{l} \binom{r}{2} \quad \binom{m}{2} \\ N \text{ rows} \end{array} \quad (32)$$

mrN columns

(Henceforth, D will be interpreted as an m th order D -matrix unless otherwise specified; thus, the subscript m is dropped.) The rank of \mathcal{D} is $(r-1)(m-1)N$; hence, \mathcal{D} fully spans the vector space orthogonal to B and therefore, the operation on H by \mathcal{D} fully retains the information content in H .

The operation $\mathcal{D}H$ converts the undifferenced information matrix H , consisting of mrN rows, into a fully redundant double differenced information matrix, \tilde{H} , consisting of $\binom{m}{2} \binom{r}{2} N$ rows. In the example of Eq. (2) with $r = 4$ and $m = 3$, the operation $\mathcal{D}H$ yields 18 row blocks of the form

$$[\mathcal{D}H]^T = \left[\begin{array}{c} \left[\begin{array}{c|c|c} (A_1 - A_2 - B_1 + B_2)^T & (A_1 - A_2 - C_1 + C_2)^T & (A_1 - A_2 - D_1 + D_2)^T \\ (B_1 - B_2 - C_1 + C_2)^T & (B_1 - B_2 - D_1 + D_2)^T & (C_1 - C_2 - D_1 + D_2)^T \\ (A_1 - A_3 - B_1 + B_3)^T & \cdot & (C_1 - C_3 - D_1 + D_3)^T \\ (A_2 - A_3 - B_2 + B_3)^T & \cdot & (C_2 - C_3 - D_2 + D_3)^T \end{array} \right] \end{array} \right] \quad (33)$$

One of the potential disadvantages of this double differencing algorithm for large m and r is clearly seen from Eq. (32) or (33); the number of rows generated by \mathcal{D} increases quadratically in both m and r . This should be compared with the Householder approach which does not expand the regression system. In a sense, the \mathcal{D} operator generates $\binom{m}{2} \binom{r}{2} - (m-1)(r-1)$ extra or "redundant" regression blocks; but, as we shall now show, these lead to a diagonal form that is equivalent to $\Lambda_{\tilde{e}}^\dagger$.

V. DIAGONAL FORMS FOR THE \mathcal{D} -MATRIX FORMULATION

From the equivalence of Eqs. (14) and (23), it follows that Π , defined by Eq. (13), is also given by

$$\Pi = \mathcal{D}^T \left[\mathcal{D} \Lambda_e \mathcal{D}^T \right]^\dagger \mathcal{D} \quad (34)$$

In the case where Λ_e is diagonal and uniform, Π can be reduced to a remarkably simple form given by

$$\Pi = \mathcal{D}^T \mathbf{W} \mathcal{D} \quad (35)$$

where \mathbf{W} is a $\binom{m}{2} \binom{r}{2} N$ by $\binom{m}{2} \binom{r}{2} N$ diagonal weighting matrix. This is equivalent to treating Λ_e as white and replacing $[\mathcal{D} \Lambda_e \mathcal{D}^T]^\dagger$ with \mathbf{W} in Eqs. (23) and (24).

This raises the question whether or not other structures for Λ_e also lead to diagonal forms. We will show that this is so and we will obtain analytic expressions for \mathbf{W} for certain cases.

The Case of Uniform Measure Error Covariance

For this case, Λ_e is given by

$$\Lambda_e = \sigma^2 \mathbf{I} \quad (36)$$

where \mathbf{I} is an $mrN \times mrN$ identity matrix. It is shown in Appendix 1 that the pseudoinverse of \mathcal{D} is given by

$$\mathcal{D}^\dagger = \frac{1}{mr} \mathcal{D}^T \quad (37)$$

For this case, Π in Eq. (34) reduces to

$$\Pi = \frac{1}{\binom{m}{2} \binom{r}{2} \sigma^2} \mathcal{D}^T \mathcal{D} \mathcal{D}^T \mathcal{D} \quad (38)$$

which, using the pseudoinverse properties of \mathcal{D} again, reduces to

$$\Pi = \frac{1}{mr\sigma^2} \mathcal{D}^T \mathcal{D} \quad (39)$$

Thus, because of the rank deficiency of $[D^T D]$, both $[D D^T]/m^2 r^2 \sigma^2$ and $I/mr\sigma^2$ serve as the pseudoinverse in Eq. (34) and provide the same result for Π . Factorization of Π in Eq. (39) leads to a form for $\Lambda_{\hat{x}}$ given by

$$\Lambda_{\hat{x}}^{-1} = \left[W^{\frac{1}{2}} D^T H \right]^T \left[W^{\frac{1}{2}} D^T H \right] \quad (40)$$

where the weighting matrix W in this case is given by

$$W = \frac{1}{mr\sigma^2} I \quad (41)$$

Also, \hat{x} is given by

$$\hat{x} = \Lambda_{\hat{x}} \left[W^{\frac{1}{2}} D^T H \right]^T \left[W^{\frac{1}{2}} D^T y \right] \quad (42)$$

An alternative derivation based on an explicit evaluation of Eq. (13) is provided in Appendix 3. It is noted that Eq. (38) provides an interesting duality property. If one thinks of Eq. (38) as the form $D^T [D D^T] D / m^2 r^2 \sigma^2$, then one has the double differencing formulation involving the inner products of arrays of dimension $\binom{m}{2} \binom{r}{2} N$. On the other hand, if one thinks of Eq. (38) having the form $\frac{1}{mr\sigma^2} [D^T D] [D D^T] \frac{1}{mr\sigma^2}$, then one has the equivalent matrices that must result from the expanded state or from the Householder approaches. Here, the array products are only mrN in dimension.

An operational definition of $\frac{1}{mr} [D^T D]$ is provided by Equation (A-14) in Appendix 2. An inspection of Eq. (A-14) shows that the operation by $\frac{1}{mr} [D^T D]$ on the matrices H and y involves the double subtraction of two mean values of the elements of these matrices from each element (plus a correction for double counting). These mean values are obtained from averaging over the set of terminals and over the set of satellites. Bierman [5] has shown that the Householder operation leading to triangularization involves the identical averaging processes in the case of uniform measurement error covariance. These averaging and subtraction processes have the net effect of cancelling the effect of clock terms in y except that it is accomplished by a linear combination of the rows of y rather than by explicit double differencing.

The Case When Λ_e Is Factorable

When Λ_e is factorable into the form

$$\Lambda_e^{1/2} = \begin{bmatrix} a^1 I_m & 0 \\ 0 & a^r I_m \end{bmatrix} \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} \quad (43)$$

where a^v is a scalar and b is an mN by mN diagonal matrix, or equivalently, when the standard deviation σ_j^v of the error for the j th terminal observing the v th satellite is given by

$$\sigma_j^v = a^v b_j \quad (44)$$

then it may be shown using techniques described in Appendices 2 and 3 that a diagonal form for W is given by

$$W = \|\Lambda_e^{-1}\|^{-2} \begin{bmatrix} \begin{array}{c|c} \frac{I}{\sigma_1^1 \sigma_2^1 \sigma_1^2 \sigma_2^2} & 0 \\ \hline 0 & \frac{I}{\sigma_{m-1}^1 \sigma_m^1 \sigma_{m-1}^2 \sigma_m^2} \end{array} & \begin{array}{c} \frac{I}{\sigma_1^1 \sigma_2^1 \sigma_1^3 \sigma_2^3} \\ \vdots \\ \frac{I}{\sigma_{m-1}^1 \sigma_m^1 \sigma_{m-1}^3 \sigma_m^3} \end{array} \\ \hline & \begin{array}{c} \frac{I}{\sigma_1^{r-1} \sigma_2^{r-1} \sigma_1^r \sigma_2^r} \\ \vdots \\ \frac{I}{\sigma_{m-1}^{r-1} \sigma_m^{r-1} \sigma_{m-1}^r \sigma_m^r} \end{array} \end{bmatrix} \quad (45)$$

(m/2) N rows

Eq. (45) is also obtained by renormalizing the clock parameters. These rescaling factors provide $m+r-1$ degrees of freedom, which are sufficient to transform this problem into an equivalent uniform one. For the factorable case it may be shown that Π has the form

$$\Pi = \mathcal{D}^T \mathbf{W} \mathcal{D} = \Lambda_e^{-1} - \Lambda_e^{-1} \mathbf{1} \mathbf{1}^T \Lambda_e^{-1} / \|\Lambda_e^{-\frac{1}{2}}\|^2 \quad (46)$$

where $\mathbf{1}$ is the N by mN unitary column matrix (see Appendix 1) and $\|\Lambda_e^{-\frac{1}{2}}\|$ the Euclidean norm of $\Lambda_e^{-\frac{1}{2}}$. Eq. (45) is not valid when Λ_e has a general diagonal form.

The success in deriving a simple analytical expression for the diagonal form of \mathbf{W} in the cases of uniform and factorable measurement error covariances encourages one to seek other cases where this is so. Wu [6] has shown for problems involving single differenced observables with one independent clock error sequence for each data stream, for example, VLBI-like sequences, that an analogous diagonal form for this single differenced version always can be established when Λ_e is diagonal. In the double differenced case, the matter is less straightforward.

VI. THE EXISTENCE OF DIAGONAL FORMS FOR THE \mathcal{D} -MATRIX FORMULATION

To explore the existence question of possible diagonal forms for \mathbf{W} one equates the forms for Π as given in Eq. (13) and in Eq. (35); from this one derives the appropriate form for \mathbf{W} . For certain forms for Λ_e one can actually carry out the matrix algebra operations indicated in Eq. (13) (See Appendix 3).

Alternatively, one can use the idempotent-related properties of Π given in Eqs. (16) and (17) (see Appendix 4). In the more general case Π would be generated numerically using Eq. (13); the numerical values of the elements of \mathbf{W} could still be obtained from equating Eqs. (13) and (35). However, any efficiencies gained in using the fully redundant approach would be lost in the numerical inversion of Eq. (13). From Eq. (13) Π may be written in the form

$$\Pi = \left[\begin{array}{c|c|c|c} A & B & C & E \\ \hline B^T & F & G & H \\ \hline C^T & G^T & I & J \\ \hline E^T & H^T & J^T & K \end{array} \right] \quad (47)$$

where each of these mN by mN sub-matrix blocks of Π is constrained by

$$\Pi \mathbf{B}^T = \Pi \mathbf{B} = \mathbf{0} \quad (48)$$

which constitutes $[m^2 r^2 N^2 - (m-1)^2 (r-1)^2] N^2$ independent orthogonality conditions.

Arbitrary forms for Λ_e break the symmetries in the off-diagonal sub-blocks of Π . Although Π is symmetric, its individual off-diagonal submatrix blocks are in general not symmetric. It would follow that \mathbf{W} could not be diagonal. However, because of the rank deficiency of $[D^T \mathbf{W} D]$, the form of \mathbf{W} is not unique, as was already demonstrated in the uniform case. Moreover, inasmuch as $H^T \Pi H$ is an array of quadratic forms, Π can be reformatted without altering the values of the elements in $H^T \Pi H$. This is accomplished by expressing each sub-block of Π as the sum of a symmetric matrix and an antisymmetric one. For example, the matrix E in Eq. (47) may be written as

$$E = \tilde{E} + \hat{E} \quad (49)$$

where \tilde{E} and \hat{E} are respectively the symmetric and antisymmetric components of E , with

$$\tilde{E} = (E + E^T)/2 \quad (50)$$

and

$$\hat{E} = (E - E^T)/2 \quad (51)$$

With these replacements Π may be written as

$$\Pi = \tilde{\Pi} + \hat{\Pi} \quad (52)$$

where $\tilde{\Pi}$ and $\hat{\Pi}$ are symmetric. The forms of these matrices are given by

$$\tilde{\Pi} = \begin{bmatrix} A & \tilde{B} & \tilde{C} & \tilde{E} \\ -\tilde{B} & F & \tilde{G} & \tilde{H} \\ -\tilde{C} & -\tilde{G} & I & \tilde{J} \\ -\tilde{E} & -\tilde{H} & -\tilde{J} & K \end{bmatrix} \quad (53)$$

and

$$\hat{\Pi} = \begin{bmatrix} 0 & \hat{B} & \hat{C} & \hat{E} \\ -\hat{B} & 0 & \hat{G} & \hat{H} \\ -\hat{C} & -\hat{G} & 0 & \hat{J} \\ -\hat{E} & -\hat{H} & -\hat{J} & 0 \end{bmatrix} \quad (54)$$

All of the r^2 sub-blocks of $\tilde{\Pi}$ are symmetric. Each of the off-diagonal sub-blocks of $\hat{\Pi}$ is antisymmetric and the negative of the corresponding sub-block in the transpose position of $\hat{\Pi}$. Hence, the diagonal sub-blocks of $\hat{\Pi}$ are zero.

In a similar fashion we write W in the form

$$W = \tilde{W} + \hat{W} \quad (55)$$

Here, \tilde{W} is a diagonal matrix and \hat{W} is a symmetric but singly striped off-diagonal matrix. These matrices are given in block diagonal form by

$$\tilde{W} = \begin{bmatrix} \tilde{W}^1 & & 0 \\ & \ddots & \\ 0 & & \tilde{W}^{(r)} \end{bmatrix} \quad (56)$$

where \tilde{W}^k is the k th diagonal submatrix of dimension $(\frac{m}{2})N$ by $(\frac{m}{2})N$. The matrix \hat{W} has the striped form

$$\hat{W} = \begin{bmatrix} 0 & \tilde{W}^2 & & 0 \\ -\tilde{W}^2 & 0 & & \\ & \ddots & \ddots & \\ 0 & & -\tilde{W}^{(r)} & 0 \end{bmatrix} \quad (57)$$

where \tilde{W}^k is antisymmetric. The form $[D^T \tilde{W} D]$ leads to r^2 mN by mN submatrix blocks that fill the entire mrN by mrN array. By Eq. (32), $[D^T \tilde{W} D]$ has the form

$$[D^T \tilde{W} D] = \begin{bmatrix} D^T (\tilde{W}^1 + \dots + \tilde{W}^{r-1}) D & -D^T \tilde{W}^1 D & -D^T \tilde{W}^2 D \dots & -D^T \tilde{W}^{r-1} D \\ -D^T \tilde{W}^1 D & D^T (\tilde{W}^1 + \tilde{W}^2 + \dots + \tilde{W}^{2r-3}) D \dots & & -D^T \tilde{W}^{2r-3} D \\ \vdots & & & \vdots \\ -D^T \tilde{W}^{r-1} D & & & \end{bmatrix} \quad (58)$$

Similarly, $[D^T \hat{W} D]$ has the form

$$[D^T \hat{W} D] = \begin{bmatrix} 0 & D^T \tilde{W}^1 D & \dots & D^T \tilde{W}^{r-1} D \\ -D^T \tilde{W}^1 D & 0 & & \\ \vdots & & \ddots & \\ \vdots & & 0 & D^T \tilde{W}^{(r)} D \\ -D^T \tilde{W}^{r-1} D & & -D^T \tilde{W}^{(r)} D & 0 \end{bmatrix} \quad (59)$$

where the matrix \mathbf{M}^k is a combination of the $\hat{\mathbf{W}}^k$'s and hence antisymmetric.

The matrix Π as given in Eq. (47) is easily obtained from the numerical matrix operations indicated by Eq. (13) with $N=1$. Upon reformatting Π according to Eqs. (50) and (54), and equating these expressions to Eqs. (58) and (59), one obtains the appropriate values for the elements of \mathbf{W} . For example, equating the upper right hand corner sub-matrix block in the two expressions for Π , one obtains

$$-\mathbf{D}^T \tilde{\mathbf{W}}^{r-1} \mathbf{D} = \tilde{\mathbf{E}} \quad (60)$$

In this expression $\tilde{\mathbf{E}}$ is the symmetric $m \times m$ matrix occupying the upper right corner block of $\tilde{\Pi}$. From the definitions of \mathbf{D} and $\tilde{\mathbf{W}}$ given by Eq. (31) and Eq. (56), Eq. (60) may be written as

$$\begin{bmatrix} (\tilde{\mathbf{W}}_1^{r-1} \dots \tilde{\mathbf{W}}_{m-1}^{r-1}) & -\tilde{\mathbf{W}}_1^{r-1} & \dots & -\tilde{\mathbf{W}}_{m-1}^{r-1} \\ -\tilde{\mathbf{W}}_1^{r-1} (\tilde{\mathbf{W}}_1^{r-1} + \tilde{\mathbf{W}}_m^{r-1} + \dots + \tilde{\mathbf{W}}_{2m-3}^{r-1}) & \dots & \dots & -\tilde{\mathbf{W}}_{2m-3}^{r-1} \\ \vdots & & \ddots & \vdots \\ -\tilde{\mathbf{W}}_{m-1}^{r-1} & -\tilde{\mathbf{W}}_{2m-3}^{r-1} & & \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{E}}_{11} & \dots & \dots & \tilde{\mathbf{E}}_{1m} \\ \vdots & & \ddots & \vdots \\ \tilde{\mathbf{E}}_{1m} & \dots & \dots & \tilde{\mathbf{E}}_{mm} \end{bmatrix} \quad (61)$$

From Eq. (61) one obtains numerical values for the elements of $\tilde{\mathbf{W}}^{r-1}$ for arbitrary forms for Λ_e . Analogous equating operations yields the antisymmetric elements of $\tilde{\mathbf{W}}$. In practice, the standard deviations of the measurement errors usually are uniform over the data streams. Nevertheless, the method described here for generating the weights is straightforward and independent of N ; the computation of Π needs to be made only once when m, r , and Λ_e are specified.

In summary, \mathbf{W} can not in general be a diagonal array, although, it can be written as the sum of a diagonal array and a singly striped symmetric array with antisymmetric submatrices. The question arises whether or not the least squares weighting matrix for the minimally redundant operator given in Eq. (29) also has an equivalent diagonal form. However, for this to be true the rows of Eq. (29) would have to be the eigenvectors associated with the non-zero eigenvalues of Λ_e , which is generally not true.

VII. THE CASE OF OUTAGES

In the usual tracking situation, satellites rise and set and the mix of satellites and terminals in the data streams changes with time. Also, receivers occasionally lose data because of instrumental or environmental reasons. These cause a considerable sorting problem for the double differencing approach. One approach is to decompose the tracking session into subsessions with epoch boundaries marked by the onset or termination of an outage for a particular terminal. These subsessions would be mutually exclusive and exhaustive. Enumerating these subsessions by the index i , it is clear (provided that e is at least white across all subsession boundaries) that the covariance matrix for the entire tracking session would be given by

$$\hat{\Lambda}_{\hat{x}}^{-1} = \sum_{i=1}^n \hat{\Lambda}_{\hat{x}}^{-1}(i) \quad (62)$$

where $\hat{\Lambda}_{\hat{x}}(i)$ is the covariance for the i th subsession and n is the total number of subsessions. By this means, whatever outages there are among the m data streams will remain invariant during each subsession.

The outage case for the j th terminal observing the v th satellite can be treated by setting $\sigma_j^v = \infty$ in Λ_e . For the case of uniform measurement error statistics, other than the ones corresponding to the missing data streams, analytic diagonal forms for \mathbf{W} can be found, although these expressions become rather complicated as the topology of Λ_e becomes more complex. When the number of outages increases sufficiently to reduce a particular terminal down to observations of only one satellite (or observations of a particular satellite by only one terminal), then the form of \mathbf{W} eliminates that terminal (satellite) from the solution set and the problem is effectively reduced in dimension from m to $m-1$ (or r to $r-1$). This is, of course, expected from our initial assumptions about the stochastic properties of the clocks.

Some examples of \mathbf{W} for outage cases are given below. For the case of a single satellite not observed by n on different terminals we set $\sigma_1^1 = \sigma_2^1 = \dots = \sigma_n^1 = \sigma/\lambda$ and assume a uniform standard deviation of σ for the remainder of the data streams. A diagonal weighting matrix is obtained (see Appendix 4) in the form given in Eq. (56) by the following expressions:

$$\mathbf{W}^v = \begin{cases} \mathbf{W} & , \quad v = 1, \dots, r-1 \\ (\mathbf{W} + \mathbf{I}/m)/(r-1), & v = r, \dots, \binom{r}{2} \end{cases} \quad (63)$$

where \mathbf{W} , a $\binom{m}{2}N$ by $\binom{m}{2}N$ diagonal matrix, is given by

$$\mathbf{W} = \frac{1}{\sigma^2} \begin{array}{c|c|c|c|c|c|c} \overbrace{(n-1)N} & \overbrace{(m-n)N} & \overbrace{(n-2)N} & \overbrace{(m-n)N} & \overbrace{N} & \overbrace{2(m-n)N} & \overbrace{\binom{m-n}{2}N} \\ \hline \begin{array}{c} aI \\ \cdot \\ aI \end{array} & & & & & & \\ \hline & \begin{array}{c} bI \\ \cdot \\ bI \end{array} & & & & & \\ \hline & & \begin{array}{c} aI \\ \cdot \\ aI \end{array} & & & & \\ \hline & & & \begin{array}{c} bI \\ \cdot \\ bI \end{array} & & & \\ \hline & & & & \begin{array}{c} aI \\ \cdot \\ aI \end{array} & & \\ \hline & & & & & \begin{array}{c} bI \\ \cdot \\ bI \end{array} & \\ \hline & & & & & & \begin{array}{c} cI \\ \cdot \\ cI \end{array} \end{array} \quad (64)$$

(64)

Here, a, b , and c are given by

$$a = \frac{r \lambda^4}{[rn\lambda^2 + (r-1+\lambda^2)(m-n)](r-1+\lambda^2)}$$

$$b = \frac{\lambda^2}{rn\lambda^2 + (r-1+\lambda^2)(m-n)}$$

$$c = \frac{r-1 + \lambda^2}{nr^2\lambda^2 + (r-1+\lambda^2)(m-n)r}$$

(65)

Setting $\lambda = 0$ in these expressions yields the weights for the case of n outages involving a single satellite and no outages on the $r-1$ remaining satellites.

In the case of two satellites with outages one must consider whether these outages involve common terminals, different terminals, or some combination. The forms for \mathbf{W} are substantially different for the three cases. The diagonal weights for these cases are provided in Appendix 4. In general, as the number of terminals experiencing outages increases, the topology of \mathbf{W} and $\mathbf{\Pi}$ becomes increasingly more complex resulting in very complicated algebraic expressions for \mathbf{W} .

VIII. CONCLUSIONS

The reader has no doubt detected a preference for linear combination methods such as the Householder method rather than double differencing. In the planned global GPS tracking network for a demonstration on NASA's planned Ocean Topography Experiment mission, TOPEX, [7], 18 GPS satellites would be aloft and roughly a dozen globally distributed GPS terminals, including one on TOPEX, would be tracking, although not all simultaneously. Nevertheless, $\binom{r}{2}\binom{m}{2}N$ is a potentially explosive number. On the other hand, double differencing has a number of attractive features that have been noted. It is of interest to compare the relative computational times of the two approaches.

Using the Householder approach it may be shown [8] that the computational time T_H , required to transform Eq. (1) into Eq. (7) at a single epoch ($N=1$) is given approximately by the expression

$$T_H = \mu(mr)(m+r+p+1)^2 \quad (66)$$

where μ is a constant coefficient for a specified computer of fixed configuration; it accounts for the detailed arithmetic and normalization operations in a floating point operation and the transfer of data in and out of memory. In the case of fully redundant double differencing one must first form the weighted double differenced regression equations from the undifferenced regression system and then apply a filter routine. For ease of comparison of

the computation times we assume that the Householder transformations are used on the weighted double difference array to obtain the triangular array of Eq. (7). In this case the computational time, including the double differencing operations, is given by

$$T_D = \mu \binom{m}{2} \binom{r}{2} (p+1) (\lambda/\mu + p+1) \quad (67)$$

Here, λ is a similar constant and λ/μ is of the order of unity. Normally, p will be some linear function of m and r . One can compare T_H and T_D for various combinations of these parameters using Eq. (66) and (67) or using actual measurements of computation time. In most cases except for modest values (m or $r \lesssim 4$) of these parameters the straight Householder approach holds a definite computational advantage, typically amounting to a factor of three to five. However, the actual computational time for these operations using either method is not usually a major concern and is a small fraction of the computational time used in data management, editing and validation activities, and in forming the regression system itself. Of more concern is the size of the arrays that occur in the double differencing approach when the fully redundant differencing approach is used.

An alternative approach is to use the algebraic form for W (when available) to generate the matrix operator Π . In this case Π is given by $[D^T W D]$ instead of by $D^T [D \Lambda D]^+ D$ thereby avoiding the numerical inversion of the double differenced measurement error covariance that results from using a less than fully redundant double difference operator. The computation time for the former approach increases as $(mr)^2$, whereas it increases at a rate somewhat higher than $(mr)^3$ for the latter approach. However, when Π is explicitly used to operate on the undifferenced regression system, one dispenses with explicit double differencing in favor of an orthogonal linear combination approach but one saves computational time and avoids the large arrays.

Linear combination methods effected through the Householder or similar transformations seem to offer a more attractive approach because they deal with only mrN regression rows and readily include the outage cases, and because they allow for ancillary stochastic modeling of the clock processes. Both the Householder approach and double differencing approaches are currently being followed at JPL [5].

If fully redundant explicit double differencing is the chosen approach, the following summarizes the procedures (using a batch mode) to obtain a state vector estimate and its covariance, which are equivalent to those obtained from linear combination methods:

- (1) Reformat the system of regression equations according to Eq. (3).
- (2) Divide the tracking session into a mutually exclusive and exhaustive set of subsessions with the outages remaining invariant within each subsession. Each subsession would be characterized by an index i and by: m_i , the number of participating terminals; r_i , the number of participating satellites; and N_i , the number of simultaneous observations by each participant.
- (3) For each of the n subsessions generate the appropriate weighting matrix W depending on the form of Λ_e , m_i and r_i .
- (4) For each subsession operate on the corresponding regression system with \mathcal{D} to generate an expanded but double differenced set of the arrays regression equations and form $W^{\frac{1}{2}}[\mathcal{D}H]$ and $W^{\frac{1}{2}}[\mathcal{D}y]$.
- (5) In the batch mode assemble the weighted and double differenced regression systems for all n subsessions into a single ensemble.
- (6) Invert this ensemble to obtain the least squares estimate of x and its covariance using one's preferred inversion method such as Householder, Cholesky, least squares, etc.

APPENDIX 1: ANALYTIC FORMS FOR $\Lambda_{\tilde{e}}^{-1}$ FOR THE MINIMALLY REDUNDANT DOUBLE DIFFERENCE OPERATOR

Analytic forms for $\Lambda_{\tilde{e}}^{-1}$ are available for the minimally redundant double difference operator when Λ_e is factorable into the form given by Eq. (43) or (44). In the case where \mathcal{D} is the minimally redundant double difference operator defined in Eqs. (29) and (30), it may be shown that $[\mathcal{D}\Lambda_e\mathcal{D}^T]^{-1}$ is given by

$$[\mathcal{D}\Lambda_e\mathcal{D}^T]^{-1} = \begin{bmatrix} \mathbf{Y}/b^2 & & 0 \\ & \ddots & \\ 0 & & \mathbf{Y}/b^r \end{bmatrix} - \left[\sum_{v=1}^r \frac{1}{b^v} \right] \begin{bmatrix} \frac{-1}{b^2 b^2} & \cdots & \frac{-1}{b^2 b^r} \\ \frac{\mathbf{Y}^{-1}}{b^r b^2} & \cdots & \frac{\mathbf{Y}^{-1}}{b^r b^r} \end{bmatrix} \quad (\text{A-1})$$

where the matrix \mathbf{Y} is defined by

$$\mathbf{Y} = [\mathbf{D}\mathbf{a}\mathbf{D}^T] \quad (\text{A-2})$$

and where \mathbf{Y}^{-1} is given by

$$\mathbf{Y}^{-1} = \begin{bmatrix} 1/a_2 & & 0 \\ & \ddots & \\ 0 & & 1/a_m \end{bmatrix} - \left[\sum_{j=1}^m \frac{1}{a_j} \right] \begin{bmatrix} \frac{1}{a_1 a_1} & \cdots & \frac{1}{a_1 a_m} \\ \vdots & & \vdots \\ \frac{1}{a_m a_1} & \cdots & \frac{1}{a_m a_m} \end{bmatrix} \quad (\text{A-3})$$

For the uniform case ($a_j = b^v = 1$), Eqn. (A-1) reduces to

$$[\mathcal{D}\Lambda_e\mathcal{D}^T]^{-1} = \frac{1}{mr} \begin{bmatrix} (r-1)\mathbf{X} & -\mathbf{X} & \cdots & -\mathbf{X} \\ -\mathbf{X} & \cdot & & \cdot \\ \vdots & & \ddots & \vdots \\ -\mathbf{X} & \cdots & & (r-1)\mathbf{X} \end{bmatrix} \quad (\text{A-4})$$

where \mathbf{X} is given by

$$\mathbf{X} = m\mathbf{I}_{m-1}^{-1} \mathbf{1}_{m-1} \mathbf{1}_{m-1}^T \quad (\text{A-5})$$

and where \mathbf{I} and $\mathbf{1}$ are defined by Eqs. (5) and (6).

APPENDIX 2: PROPERTIES OF THE FULLY REDUNDANT DOUBLE DIFFERENCE OPERATOR

The single difference matrix operator, D_m , defined in Eq. (31), can be alternatively defined by the recursive relationship

$$D_{m+1} = \left[\begin{array}{c|c} I_m & -I_m \\ \hline 0 & D_m \end{array} \right] \quad (A-6)$$

with $D_1 \equiv 0$ and where I_m and I_m are defined by Eqs. (5) and (6). The rank of D_m is $(m-1)N$. It follows from Eq. (A-6) that

$$D_m^T D_m = mI_m - \mathbf{1}_m \mathbf{1}_m^T \quad (A-7)$$

which can be shown to have $(m-1)N$ identical eigenvalues of value m and N null eigenvalues. Using the property

$$D_m \mathbf{1}_m \equiv 0 \quad (A-8)$$

it follows from Eq. (A-7) that the pseudoinverse of D_m is given by

$$D_m^\dagger = \frac{1}{m} D_m^T \quad (A-9)$$

Here, the pseudoinverse of M has the standard definition

$$\left. \begin{aligned} M^\dagger M M^\dagger &= M^\dagger \\ M M^\dagger M &= M \\ M M^\dagger &= (M^\dagger)^T M^T \\ M^\dagger M &= M^T (M^\dagger)^T \end{aligned} \right\} \quad (A-10)$$

A property of D for $n \leq m$, given by the relationship

$$\frac{1}{mn} \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & D_{m m}^T D_m \end{array} \right] \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & D_{n n}^T D_n \end{array} \right] = \frac{1}{n} \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & D_{n n}^T D_n \end{array} \right] \quad (A-11)$$

is useful in "spectral decomposition" analyses; for example, a sub-matrix block of Π in certain cases may be expressed as

$$A = \sum_{k=2}^m \alpha_k \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & D_{k k}^T D_k \end{array} \right] \Bigg\}_k^{m-k} \quad (A-12)$$

The double difference operator \mathcal{D} is defined by Eq. (32). The inner product $\mathcal{D}^T \mathcal{D}$, is given by

$$\mathcal{D}^T \mathcal{D} = r \left[\begin{array}{cc} D^T D & 0 \\ & \ddots \\ 0 & D^T D \end{array} \right] - \left[\begin{array}{cc} D^T D & \cdots D^T D \\ & \ddots \\ D^T D & D^T D \end{array} \right] \quad (A-13)$$

or alternatively, using Eq. (A-7), it may be expressed as

$$\mathcal{D}^T \mathcal{D} = m r I_{mr} - r \left[\begin{array}{cc} 1_m 1_m^T & 0 \\ & \ddots \\ 0 & 1_m 1_m^T \end{array} \right] - m \left[\begin{array}{cc} I_m & 0 \\ & \ddots \\ 0 & I_m \end{array} \right] + \left[\begin{array}{cc} 1_m 1_m^T & \cdots 1_m 1_m^T \\ & \ddots \\ 1_m 1_m^T & 1_m 1_m^T \end{array} \right] \quad (A-14)$$

It follows that the rank of $\mathcal{D}^T \mathcal{D}$ is $(m-1)(r-1)N$. From Eqs. (A-9), (31), and (A-13) it follows that the pseudoinverse of \mathcal{D} is given by

$$\mathcal{D}^\dagger = \frac{1}{mr} \mathcal{D}^T \quad (A-15)$$

Also, $\mathcal{D}^T \mathcal{D} / mr$ has $(m-1)(r-1)N$ unit eigenvalues and $(m+r-1)N$ null eigenvalues.

APPENDIX 3: THE STRUCTURE OF THE Π -MATRIX FOR A UNIFORM WEIGHTING MATRIX

To obtain insight into the structure of Π in double differencing, it is helpful to examine the structure of Π that follows from its direct computation using Eq. (13) for the case of uniform measurement error covariance. To examine Π further, we will need the pseudoinverse of $[B^T \Lambda_e^{-1} B]$. Alternately, one could strike one of the columns of B , thereby eliminating the rank deficiency in B , and deal with the inverse of $[B^T \Lambda_e^{-1} B]$ instead of the pseudoinverse. Column striking spoils the symmetry of the problem as posed, although it leads to the same results as the pseudoinverse applied to the fully dimensioned B . Either way, $[B^T \Lambda_e^{-1} B]$ is difficult to invert except in special cases where Λ_e is factorable, outages, etc. Let us show the form of $[B^T \Lambda_e^{-1} B]$ when Λ_e is uniform. It can be shown for this case that

$$B^T \Lambda_e^{-1} B = \frac{1}{\sigma^2} \left[\begin{array}{c|c} r I_m & -1_m 1_r^T \\ \hline -1_r 1_m^T & m I_r \end{array} \right] \quad (A-16)$$

and that its pseudoinverse is given by

$$[B^T \Lambda_e^{-1} B]^\dagger = \frac{\sigma^2}{mr(m+r)^2} \left[\begin{array}{c|c} m(m+r)^2 I_m - m(m+2r) 1_m 1_m^T & -mr 1_m 1_r^T \\ \hline -mr 1_r 1_m^T & r(m+r)^2 I_r - r(r+2m) 1_r 1_r^T \end{array} \right] \quad (A-17)$$

After some matrix algebra it may be shown using Eq. (13) that Π becomes

$$\Pi = \frac{\sigma^2}{mr} \left[\begin{array}{c|c|c|c} (r-1)(m I_m - 1_m 1_m^T) & -m I_m + 1_m 1_m^T & \dots & -m I_m + 1_m 1_m^T \\ \hline -m I_m + 1_m 1_m^T & (r-1)(m I_m - 1_m 1_m^T) & \dots & -m I_m + 1_m 1_m^T \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline -m I_m + 1_m 1_m^T & -m I_m + 1_m 1_m^T & \dots & (r-1)(m I_m - 1_m 1_m^T) \end{array} \right] \quad (A-18)$$

But, using Eq. (A-7) we now identify the array in Eq. (A-18) with that in Eq. (A-13). In terms of the \mathcal{Q} -matrices, we conclude that Π is given by

$$\Pi = \frac{\sigma^2}{mr} \mathcal{D}^T \mathcal{D} \quad (\text{A-19})$$

Thus, the double differencing structure appears in the expanded state space array for the covariance of $\hat{\mathbf{x}}$ when it is factored according to Eq. (A-19).

One can also derive other analytical forms for \mathbf{W} from the expanded state space approach whenever explicit forms for the submatrix elements of Π can be obtained analytically from Eq. (13). Another approach using the idempotent properties of $\Lambda_e^{\frac{1}{2}} \Pi \Lambda_e^{\frac{1}{2}}$, is discussed in Appendix 4.

APPENDIX 4: OTHER DIAGONAL WEIGHTING MATRICES FOR MULTIPLE OUTAGES

Inasmuch as the major topological properties of Π are invariant to N , we set $N=1$ in the discussion to follow without significant loss of generality. Another useful way to arrive at explicit forms for Π is to use the relationship

$$\Pi = \Pi \Lambda_e \Pi \quad (\text{A-20})$$

which follows from Eq. (13); also

$$\text{Trace} \left[\Lambda_e^{\frac{1}{2}} \Pi \Lambda_e^{\frac{1}{2}} \right] = (m-1)(r-1) \quad (\text{A-21})$$

In addition, Π satisfies the $m^2 r^2 - (m-1)^2 (r-1)^2$ orthogonality conditions defined by

$$\Pi B = B^T \Pi \equiv 0 \quad (\text{A-22})$$

In theory these conditions are sufficient to solve for Π when the form for Λ_e is given. In practice, the topology of Π , except for the simplest of forms for Λ_e , becomes so complex that the resulting explicit algebraic expressions for the elements of Π in terms of m, r and the elements of Λ_e are rather tedious to evaluate. In these cases a more practical approach is to numerically invert Π for $N=1$ as given by Eq. (13).

As examples of this analytical technique the single and double outage cases are presented. For a single satellite outage we consider a Λ_e with the form

$$\Lambda_e = \sigma^2 \left[\begin{array}{c|c} \begin{matrix} \lambda^{-2} & & 0 \\ & \ddots & \\ 0 & & \lambda^{-2} \end{matrix} & \begin{matrix} 0 \\ \\ 0 \end{matrix} \\ \hline \begin{matrix} 0 \\ \\ 0 \end{matrix} & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \end{array} \right] \left. \begin{array}{l} \left. \vphantom{\begin{matrix} \lambda^{-2} \\ \ddots \\ \lambda^{-2} \end{matrix}} \right\} n \\ \left. \vphantom{\begin{matrix} 1 \\ \ddots \\ 1 \end{matrix}} \right\} m-n \end{array} \right] \quad (\text{A-23})$$

where $\lambda = 0$ corresponds to the case of n terminals not observing a single particular satellite with the remainder of the terminals observing all r satellites. Applying part of the conditions in Eq. (A-22) one obtains a Π of the form

$$\Pi = \begin{bmatrix} (r-1)A & -A & . & . & . & . & . & . & -A \\ -A & A+(r-2)B & -B & . & . & . & . & . & -B \\ . & -B & . & . & . & . & . & . & A+(r-2)B \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ -A & -B & . & . & . & . & . & . & A+(r-2)B \end{bmatrix} \quad (A-24)$$

Applying Eq. (A-20) one obtains the conditions on A and B

$$rA^2 = A \quad (A-25)$$

and

$$(r-1)B = [D^T D / m - A] \quad (A-26)$$

Next, equating the elements of the matrices in these expressions and applying Eq. (A-21) and the remainder of the orthogonality conditions one obtains for A the form

$$A = \left\{ \begin{array}{c} \left[\begin{array}{c|c} \frac{eI+a11^T}{b11^T} & \frac{b11^T}{dI+c11^T} \end{array} \right] \left. \begin{array}{l} n \\ m-n \end{array} \right\} \quad (A-27)$$

where the coefficients a, b , and c are given in Eq. (65), and d and r are given by.

$$\begin{aligned} e &= \lambda^2 / (r-1+\lambda^2) \\ d &= 1/r \end{aligned} \quad (A-28)$$

The elements of B follow from Eq. (A-26). Note that no symmetricizing of A and B is necessary because they are already symmetric for this special case. With these expressions for A and B placed in Π as given by Eq. (A-24) which is in turn equated to $[D^T W D]$ as given by Eqs. (58) and (61), one obtains the algebraic expressions for the diagonal weights that are given in Eq. (65).

For the double outage case we let one satellite incur outages on x different terminals. We let a second satellite incur y outages that involve terminals among those x terminals not observing the first satellite. We let the second

satellite also incur z outages involving terminals observing the first satellite. Thus, the second satellite incurs a total of $y+z$ outages with $y \leq x$ and $z \leq m-x$. (There is a duality symmetry in this problem that allows one to exchange the roles of satellites and terminals.) The form for Λ_e may be given by

$$\Lambda_e^{-1} = \sigma^{-2} \begin{bmatrix} 0 \dots 0 & & & & 0 \\ & 1 \dots & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 0 \dots 0 \\ & & & & & 1 \dots 1 \\ & 0 & & & & & 0 \dots 0 \end{bmatrix} \begin{matrix} \left. \begin{matrix} \\ \\ \\ \\ \end{matrix} \right\} x \\ \left. \begin{matrix} \\ \\ \\ \end{matrix} \right\} m-x+m(r-2) \\ \left. \begin{matrix} \\ \\ \end{matrix} \right\} y \\ \left. \begin{matrix} \\ \end{matrix} \right\} m-y-z \\ \left. \begin{matrix} \\ \end{matrix} \right\} z \end{matrix} \quad (A-29)$$

Following the same routine as used in the previous example, it may be shown that the topology of Π has the form

$$\begin{bmatrix} (r-2)B+A & -B & -B & -B & -A \\ -B^T & B+E+(r-3)C & -C & -C & -E^T \\ -B^T & -C & & & \cdot \\ \cdot & \cdot & \ddots & & \cdot \\ \cdot & \cdot & & -C & \\ -B^T & -C & -C & & -E^T \\ -A^T & -E & & -E & A+(r-2)E^T \end{bmatrix} \quad (A-30)$$

Where each of the sub-matrices A, B, C and E are m by m . Here, the x missing data streams for the first satellite correspond to zeros in the first x rows and columns of Π . Similarly, the z missing data streams for the second satellite (which for convenience has been placed last) correspond to zeros in the last z rows and columns. The y outages of the second satellite correspond to zeros in rows and columns numbering $(r-1)m+1$ through $(r-1)m+y$. Thus B and E , the top and bottom edge matrices are of rank $m-x-1$ and $m-y-z-1$, respectively. The rank of A , the corner matrix, is the smaller of the ranks of A and E . These matrices are

generally not symmetric and will have to be decomposed into symmetric and antisymmetric parts before equating them to the diagonal weight matrices given in Eqs. (58) and (59). The core matrix C , is symmetric and explicitly appears whenever $r \geq 4$. For $r=2$ or 3 , C may be considered as a virtual matrix which does not explicitly appear in Π . For all values of $r \geq 2$ it may be shown upon applying Eqn. (A-20) that B, C and E satisfy the relation

$$(r-2)C = D^T D/m - B - E \quad (A-31)$$

The topologies of B, C and E can be described in block matrix form by

$$B = \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline B_{31} & B_{32} & B_{33} & B_{34} \\ \hline B_{41} & B_{42} & B_{43} & B_{44} \end{array} \right] \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} \begin{array}{l} x \\ m-x-z \\ z \end{array} \quad (A-32)$$

$$C = \left[\begin{array}{c|c|c|c} C_{11} & C_{12} & C_{13} & C_{14} \\ \hline C_{12} & C_{22} & C_{23} & C_{24} \\ \hline C_{13} & C_{23} & C_{33} & C_{34} \\ \hline C_{14} & C_{24} & C_{34} & C_{44} \end{array} \right] \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} \begin{array}{l} y \\ x-y \\ m-x-z \\ z \end{array} \quad (A-33)$$

and

$$E = \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline E_{21} & E_{22} & E_{23} & E_{24} \\ \hline E_{31} & E_{32} & E_{33} & R_{34} \\ \hline 0 & 0 & 0 & 0 \end{array} \right] \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} \begin{array}{l} y \\ y-x \\ m-x-z \\ z \end{array} \quad (A-34)$$

It may be shown that the non-zero elements of B satisfy the relation

$$b_{ij} = c_{ij} + \sum_{k=1}^x c_{kj} / (m-x) \quad (A-35)$$

Similarly, the non-zero elements of E are given by

$$e_{ij} = c_{ij} + \left[\sum_{k=1}^y c_{kj} + \sum_{k=m-z+1}^m c_{kj} \right] / (m-y-z) \quad (A-36)$$

Also, the non-zero elements of A are given by

$$a_{ij} = b_{ij} + \sum_{k=m-z+1}^m b_{ik} / (m-y-z) \quad (A-37)$$

It may be shown that the form of C has the additional properties given by

$$C = \begin{bmatrix} aI+b11^T & b11^T & b11^T & b11^T \\ b11^T & cI+d11^T & e11^T & f11^T \\ b11^T & e11^T & gI+h11^T & j11^T \\ b11^T & f11^T & j11^T & kI+l11^T \end{bmatrix} \quad (A-38)$$

Thus, it suffices to determine the elements of the core matrix C given in Eq. (A-38). Applying these relationships between B,C and E in Eqs. (A-31), (A-35) and (A-36) and the remainder of the orthogonality conditions in Eq. (A-22), one obtains

$$\left. \begin{aligned} a &= 1/(r-2) \\ c &= 1/(r-1) \\ g &= 1/r \\ k &= 1/(r-1) \\ b &= -1/m(r-2) \\ d &= \frac{-1}{m(r-1)} + \frac{(r-1)(y+z)(m-x) - z(m-x+y)}{m(r-1)(r-2)H} \\ e &= \frac{(r-2)(m-x)(rz-(r-1)m) + y[(r-1)(m-x)-z]}{m(r-2)H} \end{aligned} \right\} \quad (A-39)$$

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|---|--|--|-----------|
| 1. Report No. JPL PUB. 86-30 | 2. Government Accession No. | 3. Recipient's Catalog No. | |
| 4. Title and Subtitle A Fully Redundant Double Difference Algorithm for Obtaining Minimum Variance Estimates from GPS Observations | | 5. Report Date December 15, 1986 | |
| | | 6. Performing Organization Code | |
| 7. Author(s) William G. Melbourne | | 8. Performing Organization Report No. | |
| 9. Performing Organization Name and Address JET PROPULSION LABORATORY California Institute of Technology 4800 Oak Grove Drive Pasadena, California 91109 | | 10. Work Unit No. B-055-10-01-09-97 | |
| | | 11. Contract or Grant No. NAS7-918 | |
| | | 13. Type of Report and Period Covered | |
| 12. Sponsoring Agency Name and Address NATIONAL AERONAUTICS AND SPACE ADMINISTRATION Washington, D.C. 20546 | | | |
| | | 14. Sponsoring Agency Code | |
| 15. Supplementary Notes | | IN -04-CR 115286 | |
| 16. Abstract 438 <p>In double differencing a regression system obtained from concurrent Global Positioning System (GPS) observation sequences, one either under-samples the system to avoid introducing colored measurement statistics, or one fully samples the system incurring the resulting non-diagonal covariance matrix for the differenced measurement errors. A suboptimal estimation result will be obtained in the under-sampling case and will also be obtained in the fully sampled case unless the color noise statistics are taken into account. The latter approach requires a least squares weighting matrix derived from inversion of a non-diagonal covariance matrix for the differenced measurement errors instead of inversion of the customary diagonal one associated with white noise processes. This publication presents the so-called fully redundant double differencing algorithm for generating a weighted double differenced regression system that yields equivalent estimation results, but features for certain cases a diagonal weighting matrix even though the differenced measurement error statistics are highly colored.</p> | | | |
| 17. Key Words (Selected by Author(s)) Global Positioning System, Algorithm, Variance | | 18. Distribution Statement Unclassified-Unlimited | |
| 19. Security Classif. (of this report) Unclassified | 20. Security Classif. (of this page) Unclassified | 21. No. of Pages 41 | 22. Price |